

## STRUCTURE OF SYMMETRIC TENSORS OF TYPE $(0, 2)$ AND TENSORS OF TYPE $(1, 1)$ ON THE TANGENT BUNDLE

BY

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**ABSTRACT.** The concepts of  $M$ -tensor and  $M$ -connection on the tangent bundle  $TM$  of a smooth manifold  $M$  are used in a study of symmetric tensors of type  $(0, 2)$  and tensors of type  $(1, 1)$  on  $TM$ . The constructions make use of certain local frames adapted to an  $M$ -connection. They involve extending known results on  $TM$  using tensors on  $M$  to cases in which these tensors are replaced by  $M$ -tensors. Particular attention is devoted to (pseudo-) Riemannian metrics on  $TM$ , notably those for which the vertical distribution on  $TM$  is null or nonnull, and to the construction of almost product and almost complex structures on  $TM$ .

**1. Introduction.** Let  $M$  be a smooth manifold and  $TM$  its tangent bundle. In his 1958 paper [1], S. Sasaki constructed a Riemannian metric on  $TM$  from a Riemannian metric on  $M$ , heralding the beginning of the differential geometry of the tangent bundle. Since then, other Riemannian metrics on  $TM$  have been constructed (see Yano and Ishihara [3, Chapter IV]) but no general method of construction has emerged.

Recently, two of us (Wong and Mok [1]) introduced the concepts of  $M$ -tensor and three types of connections on  $TM$ , and used them to clarify the relationship between several related known concepts on  $TM$ . In this paper, we shall show how the concepts of  $M$ -tensor and one of these connections (which we now call  $M$ -connection) enable us to have a complete picture of the structures of the symmetric tensors of type  $(0, 2)$  and tensors of type  $(1, 1)$  on  $TM$ . We shall see also that many of the known results on  $TM$  arising from tensors on  $M$  have a meaning when these tensors are replaced by  $M$ -tensors on  $TM$ .

In §2, we fix our notations and give some formulas that will be frequently used. In §3, we show that the concepts of  $M$ -tensor and  $M$ -connection are inherent in the transformation law of the components of a symmetric tensor of type  $(0, 2)$  on  $TM$ . In §4, we consider certain local frames adapted to an  $M$ -connection, and show how any tensor on  $TM$  can be expressed in terms of an  $M$ -connection and some  $M$ -tensors of the same type. In §§5–8, we study

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the structure of a symmetric tensor of type  $(0, 2)$  on  $TM$ . Two important cases are singled out for detailed discussion: the symmetric tensors of type  $(0, 2)$  on  $TM$  with respect to which the vertical distribution is respectively null or nonnull. Virtually all the Riemannian metrics on  $TM$  that have so far appeared in the literature are of one of these types. It is interesting though not unexpected that the original Sasaki metric occupies a central position among all the possible Riemannian metrics on  $TM$  with respect to which the vertical distribution is nonnull. In §§9–11, we carry out a similar study of the structure of a tensor of type  $(1, 1)$  on  $TM$ , and in particular the structure of a tensor  $F$  of type  $(1, 1)$  satisfying the condition  $F^2 = \lambda E$ , where  $\lambda$  is a real number and  $E$  the “identity tensor” on  $TM$ . Finally in §12, we determine all the compatible Riemannian metrics and almost product (resp. almost complex) structures on  $TM$  whose associated  $M$ -tensors are everywhere nonsingular or zero.

We remark that the problem for the cotangent bundle  $T^*M$  similar to that considered in this paper for the tangent bundle  $TM$  can be solved by using the  $M$ -tensors on  $T^*M$  and a type of connection equivalent to a horizontal distribution on  $T^*M$ . Details will be given in a forthcoming paper (Wong and Mok [2]).

**2. The tangent bundle  $TM$ .** Throughout this paper, the indices  $a, b, c, \dots; h, i, j, \dots$  run over the range  $\{1, 2, \dots, n\}$ , while the indices  $\alpha, \beta, \gamma, \dots$  run over the range  $\{1, 2, \dots, n, n+1, \dots, 2n\}$ .  $\bar{h}$  will denote  $n+h$ . Summation over repeated indices is always implied.

When matrices are used, we denote their elements by  $x^i, A_{ij}$  or  $F_j^i$ . In each case,  $i$  denotes the row and  $j$  denotes the column. A matrix  $A$  whose elements are  $A_{ij}$  will be denoted by  $[A_{ij}]$ . The transpose of  $A$  is denoted by  $A^t$  and the inverse of  $A$ , if it exists, is denoted by  $A^{-1}$ . The  $n \times n$  identity matrix is denoted by  $I$ .

Let  $M$  be an  $n$ -dimensional smooth (i.e.,  $C^\infty$ ) manifold which we shall always assume to be connected and paracompact. We denote by  $T_p M$  the tangent space to  $M$  at the point  $p \in M$ , and by  $TM = \bigcup_{p \in M} T_p M$  the *tangent bundle* of  $M$  with base space  $M$ , fibers  $T_p M$  and projection  $\pi: TM \rightarrow M$  which sends the elements of  $T_p M$  to  $p$ . If  $U$  is any subset of  $M$ , we denote  $\pi^{-1}U$  by  $TU$ , so that in particular  $T_p M = \pi^{-1}(p) = Tp$ . If  $\sigma \in TM$  and  $\pi\sigma = p$ , then  $\sigma \in Tp$  and the tangent space to  $Tp$  at  $\sigma$  is an  $n$ -dimensional subspace  $V_\sigma$  of  $T_\sigma(TM)$ . The assignment  $\sigma \rightarrow V_\sigma$  is an integrable distribution of  $n$ -planes on  $TM$  which we call the *vertical distribution* on  $TM$  and denote by  $V$ .

Let  $(U, x)$  be a chart in  $M$  with neighborhood  $U$  and coordinate function  $x = [x^i]$ . If  $\sigma \in TM$ , then  $\sigma \in TU$  for some  $U$  so that  $\sigma$  is a tangent vector to  $M$  at  $p = \pi\sigma \in U$ . Suppose that  $\sigma = y^i(\partial/\partial x^i)_p$ , i.e.,  $y^i$  are the components

of  $\sigma$  in the chart  $(U, x)$  in  $M$ . Then  $(TU, (x, y))$ , where  $y = [y^i]$ , is a chart in  $TM$  which we say is induced from the chart  $(U, x)$  in  $M$ . If  $(TU', (x', y'))$  is another induced chart in  $TM$  such that  $TU \cap TU'$  is nonempty, then the restrictions of the coordinate functions  $(x, y)$  and  $(x', y')$  to  $TU \cap TU'$  are related by

$$(2.1) \quad \begin{aligned} x^{i'} &= x^i(x^1, \dots, x^n), \\ y^{i'} &= y^j p_{ji}^{i'}, \quad \text{where } p_{ji}^{i'} = \partial x^{i'} / \partial x^i. \end{aligned}$$

Here and in what follows, a dash ' indicates quantities related to  $U'$  or  $TU'$ , as the case may be. Let us denote by  $\partial$  the operator  $y^i \partial / \partial x^i$  on functions defined on  $TU$ . Then,

$$\partial p_{ji}^{i'} = y^j \frac{\partial^2 x^{i'}}{\partial x^j \partial x^i} \equiv y^j p_{ji}^{i'},$$

and differentiation of (2.1) gives

$$(2.2) \quad dx' = P dx, \quad dy' = (\partial P) dx + P dy,$$

where  $P = [p_{ji}^{i'}]$ ,  $\partial P = [\partial p_{ji}^{i'}]$ . Thus, the Jacobian matrix of the transformation (2.1) is

$$(2.3) \quad \begin{bmatrix} P & 0 \\ \partial P & P \end{bmatrix}.$$

Let  ${}^sTM$  be the subset of  $TM$  consisting of all the nonzero tangent vectors of  $M$ . Then  ${}^sTM$  is an open submanifold and also a subbundle of  $TM$  which we call the *slit tangent bundle* of  $M$ . In  ${}^sTM$  we have the induced charts  $({}^sTU, (x, y))$  which are the restriction to  ${}^sTM$  of the induced charts  $(TU, (x, y))$  in  $TM$  so that  $y \neq 0$  in  $({}^sTU, (x, y))$ . It will be clear that all the results on  $TM$  in this paper also hold for  ${}^sTM$ .

We now recall the Sasaki metric on  $TM$  (see Sasaki [1]) constructed from a Riemannian metric  $g$  on  $M$ . Let the components of  $g$  in  $(U, x)$  be  $g_{ij}$  and  $\{^i_{jk}\}$  the Christoffel symbols of  $g_{ij}$ . We denote by  $g$  also the matrix  $[g_{ij}]$  and by  $\Gamma$  the matrix  $[\Gamma^i_j]$ , where

$$(2.4) \quad \Gamma^i_j = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} y^k.$$

Then the component matrix in  $TU$  of the Sasaki metric is

$$(2.5) \quad \begin{bmatrix} g + \Gamma^i g \Gamma^i & \Gamma^i g \\ g \Gamma & g \end{bmatrix}.$$

As has been observed by several authors, (2.5) still defines a Riemannian metric on  $TM$  if  $\{^i_{jk}\}$  in (2.4) are replaced by the components  $\gamma^i_{jk}$  of a linear connection on  $M$ .

3. **The concepts of  $M$ -tensor and  $M$ -connection on  $TM$ .** In this section, we shall show how the concepts of  $M$ -tensor and  $M$ -connection on  $TM$  arise naturally from the transformation law for the components of a symmetric tensor of type  $(0, 2)$  on  $TM$ .

Let  $G$  be any symmetric  $(0, 2)$ -tensor on  $TM$  and

$$\begin{bmatrix} A & B' \\ B & c \end{bmatrix}$$

the component matrix of  $G$  in  $TU$ , where  $A, B, c$  are  $n \times n$  matrix functions of  $(x, y)$  of which  $A$  and  $c$  are symmetric. Then in  $TU \cap TU'$ , the component matrix of  $G$  in  $TU$  and that in  $TU'$  are related by

$$\begin{bmatrix} A & B' \\ B & c \end{bmatrix} = \begin{bmatrix} P' & (\partial P)' \\ 0 & P' \end{bmatrix} \begin{bmatrix} A' & B'' \\ B' & c' \end{bmatrix} \begin{bmatrix} P & 0 \\ \partial P & P \end{bmatrix},$$

which is easily seen to be equivalent to the following equations:

$$(3.1) \quad \begin{aligned} c &= P'c'P, & B &= P'(B'P + c'\partial P), \\ A &= [P'A' + (\partial P)'B']P + [P'B'' + (\partial P)'c']\partial P. \end{aligned}$$

Suppose now that  $c$  is everywhere nonsingular. Guided by the form of the Sasaki metric (2.5), we tentatively put

$$(3.2) \quad B = c\Gamma, \quad A = a + c + \Gamma'c\Gamma,$$

where  $\Gamma = [\Gamma_j^i]$ ,  $a = [a_{ij}]$  are  $n \times n$  matrices. Note that since  $c$  is nonsingular, first  $\Gamma$  and then  $a$  are uniquely determined by (3.2). If we define  $\Gamma$  and  $a$  by (3.2), the equations (3.1) imply that

$$(3.3) \quad \Gamma'P = P\Gamma - \partial P,$$

$$(3.4) \quad a = P'a'P.$$

In fact, (3.3) is obtained by substituting (3.2)<sub>1</sub> in (3.1)<sub>2</sub> and then using (3.1)<sub>1</sub>, and (3.4) is obtained by substituting (3.2) in (3.1)<sub>3</sub> and then simplifying the result by (3.1)<sub>1</sub> and (3.3). Conversely, if (3.1)<sub>1</sub>, (3.3) and (3.4) hold, and  $A, B$  are given by (3.2), then all of equations (3.1) hold.

Thus, we have proved that equations (3.1) are equivalent to equations (3.1)<sub>1</sub>, (3.3) and (3.4), the equivalence being achieved by (3.2).

Motivated by equations (3.3) and (3.1)<sub>1</sub> and (3.4), the last two of which mean that the elements of the matrices  $c$  and  $a$  (which are functions of  $(x, y)$ ) transform like the components of a  $(0, 2)$ -tensor on  $M$ , we formulate the following definitions.

**DEFINITION.** An  $M$ -connection on  $TM$  is the geometric object determined by an assignment to each induced chart  $(TU, (x, y))$  of an  $n \times n$  matrix function  $\Gamma = [\Gamma_j^i]$  such that

$$(3.5) \quad \Gamma'P = P\Gamma - \partial P \quad \text{on } TU \cap TU'.$$

We call  $\Gamma_j^i$  the *components* of the  $M$ -connection and refer to the  $M$ -connection simply as the  $M$ -connection  $\Gamma$  or  $\Gamma_j^i$ . (Similar terminology will be used for the  $M$ -tensors defined below.  $M$ -connection has been called *connection of type* (1, 1) in Wong and Mok [1].)

DEFINITION. An  $M$ -tensor of type  $(r, s)$  on  $TM$  is the geometric object determined by an assignment to each induced chart  $(TU, (x, y))$  of a set of  $n^{r+s}$  functions  $S_{j_1 \dots j_s}^{i_1 \dots i_r}(x, y)$  that behave like the components of a tensor of type  $(r, s)$  on  $M$ , i.e.,

$$(3.6) \quad S_{j_1' \dots j_s'}^{i_1' \dots i_r'} = p_{i_1'}^{i_1} \dots p_{i_r'}^{i_r} S_{j_1 \dots j_s}^{i_1 \dots i_r} p_{j_1}^{j_1'} \dots p_{j_s}^{j_s'} \quad \text{on } TU \cap TU'.$$

For reasons that will be clear in §4 (Theorem 4.2), we shall refer to the symmetric  $M$ -tensor  $c$  of type (0, 2) appearing in the discussion above as the  $M$ -tensor associated with  $G$ . The following theorem summarises what we have proved so far.

THEOREM 3.1. *The most general symmetric tensor of type (0, 2) on  $TM$  whose associated  $M$ -tensor  $c$  is everywhere nonsingular has a component matrix of the form*

$$(3.7) \quad \begin{bmatrix} a + \Gamma^c \Gamma & \Gamma^c \\ c \Gamma & c \end{bmatrix},$$

where  $\Gamma$  is an  $M$ -connection and  $a$  is a symmetric  $M$ -tensor of type (0, 2) on  $TM$ .

Similarly we can show that the concepts of  $M$ -tensor and  $M$ -connection on  $TM$  are also inherent in the transformation law for the components of a tensor of type (1, 1) on  $TM$ .

We end this section with the following two easy-to-prove theorems.

THEOREM 3.2. (a) *Let  $t = 1, \dots, r$ , let  $\lambda_t$  be  $r$  real numbers and let  $\Gamma_t$  be  $r$   $M$ -connections on  $TM$ . Then  $\sum \lambda_t \Gamma_t$  is an  $M$ -connection iff  $\sum \lambda_t = 1$ , and is an  $M$ -tensor of type (1, 1) iff  $\sum \lambda_t = 0$ . In particular, if  $\Gamma$  is any  $M$ -connection, then any other  $M$ -connection is of the form  $\Gamma + S$ , where  $S$  is some  $M$ -tensor of type (1, 1).*

(b) *If  $[\gamma_{jk}^i]$  is a linear connection on  $M$  and  $\Gamma_k^i = y^j \gamma_{jk}^i$ , then  $\Gamma = [\Gamma_j^i]$  is an  $M$ -connection on  $TM$ .*

(c) *Tensors on  $M$  can be considered as  $M$ -tensors on  $TM$ . Addition, scalar multiplication, tensor product and contraction of  $M$ -tensors on  $TM$  can be defined as for tensors on  $M$ . The space of  $M$ -tensors on  $TM$  is an algebra over the reals.*

(d) *If  $S_j^{i_1 \dots i_r}$  is an  $M$ -tensor of type  $(r, s)$ , then  $\partial^t S_j^{i_1 \dots i_r} / \partial y^{k_1} \dots y^{k_t}$  is an  $M$ -tensor of type  $(r, s + t)$ .*

(e) *Any  $M$ -tensor on  $TM$  (but not an  $M$ -tensor on  ${}^s TM$ ) determines in a*

natural way a tensor on  $M$ . In fact, if  $S$  is an  $M$ -tensor of type  $(r, s)$ , then its components  $S_{j_1 \dots j_s}^{i_1 \dots i_r}(x, y)$  in  $TU$  determine the components  $S_{j_1 \dots j_s}^{i_1 \dots i_r}(x, 0)$  in  $U$  of a tensor of type  $(r, s)$  on  $M$ .

Any  $M$ -tensor on  $TM$  also determines in a natural way a tensor on  $TM$  as stated in the following

**THEOREM 3.3.** *Let  $S_{j_1 \dots j_s}^{i_1 \dots i_r}$  be an  $M$ -tensor of type  $(r, s)$  on  $TM$ . If we put*

$$(3.8) \quad S = S_{j_1 \dots j_s}^{i_1 \dots i_r} \frac{\partial}{\partial y^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s},$$

*then  $S$  is a tensor of type  $(r, s)$  on  $TM$ . Conversely, if (3.8) defines a tensor of type  $(r, s)$  on  $TM$ , then the functions  $S_{j_1 \dots j_s}^{i_1 \dots i_r}$  are components in  $(TU, (x, y))$  of an  $M$ -tensor of type  $(r, s)$  on  $TM$ .*

**4. Adapted frames.** The  $M$ -connection we introduced in §3 has a very simple geometric interpretation which we now explain. Let  $V$  be the vertical distribution on  $TM$ . A distribution  $H$  of  $n$ -planes on  $TM$  is said to be a *horizontal distribution* if  $V$  and  $H$  are complementary, i.e.,  $T_\sigma(TM) = V_\sigma \oplus H_\sigma$  (direct sum) at each  $\sigma \in TM$ . Using the fact that the vertical distribution  $V$  is determined on each  $TU$  by the following system of Pfaffian equations:

$$(4.1) \quad \omega^i \equiv dx^i = 0,$$

we can prove (cf. also Yano and Okubo [1])

**THEOREM 4.1.** *The concept of  $M$ -connection on  $TM$  is equivalent to the concept of horizontal distribution on  $TM$ , the equivalence being achieved by the fact that the restriction of the horizontal distribution to  $TU$  is determined by the following system of Pfaffian equations:*

$$(4.2) \quad \omega^{\bar{i}} \equiv dy^i + \Gamma_j^i dx^j = 0.$$

On account of Theorem 4.1, we shall denote by  $H_\Gamma$  the horizontal distribution determined by the  $M$ -connection  $\Gamma$ .

It follows that the  $2n$  1-forms

$$[\omega^\alpha] = \begin{bmatrix} \omega^i \\ \omega^{\bar{i}} \end{bmatrix}$$

defined by (4.1) and (4.2) form a coframe on  $TU$ . The frame  $[D_\beta] = [D_j, D_{\bar{j}}]$  on  $TU$  dual to  $[\omega^\alpha]$  consists of the following  $2n$  vector fields on  $TU$ :

$$(4.3) \quad D_j = \frac{\partial}{\partial x^j} - \Gamma_j^i \frac{\partial}{\partial y^i}, \quad D_{\bar{j}} = \frac{\partial}{\partial y^j},$$

of which the  $n$  vector fields  $D_j$  span the horizontal distribution  $H_\Gamma$  and the  $n$  vector fields  $D_{\bar{j}}$  span the vertical distribution.

We call  $[\omega^\alpha]$  and  $[D_\beta]$  respectively the *coframe* and *frame* in  $TU$  adapted to

the  $M$ -connection  $\Gamma$ . If  $[\omega^\alpha], [D_\beta]$  are the coframe and frame in  $TU'$  adapted to  $\Gamma$ , then in  $TU \cap TU'$  we have

$$(4.4) \quad [\omega^{\alpha'}] = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} [\omega^\alpha], \quad [D_\beta] = [D_{\beta'}] \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}.$$

On the other hand, it follows from (4.1), (4.2) and (4.3) that the adapted frame and the natural frame on  $TU$  are related by

$$(4.5) \quad [\omega^\alpha] = N[dx^\alpha], \quad [D_\beta] = [\partial/\partial x^\beta]N^{-1},$$

where

$$N = \begin{bmatrix} I & 0 \\ \Gamma & I \end{bmatrix}, \quad N^{-1} = \begin{bmatrix} I & 0 \\ -\Gamma & I \end{bmatrix}.$$

Let  $S$  be a tensor, say of type  $(1, 2)$ , on  $TM$ . The frame components of  $S$  in  $TU$  are the functions  $S_{\beta\gamma}^\alpha$  appearing in the equation

$$S = S_{\beta\gamma}^\alpha D_\alpha \otimes \omega^\beta \otimes \omega^\gamma.$$

It follows from (4.5) that for a tensor  $G$  of type  $(0, 2)$  on  $TM$ , its frame component matrix  $\bar{G}$  is related to its component matrix (also denoted by  $G$ ) by

$$(4.6) \quad \bar{G} = (N^{-1})'GN^{-1}.$$

Similarly, for a tensor  $F$  of type  $(1, 1)$  on  $TM$ , its frame component matrix  $\bar{F}$  is related to its component matrix (also denoted by  $F$ ) by

$$(4.7) \quad \bar{F} = NFN^{-1}.$$

By means of the adapted frames and coframes, we can obtain from a tensor of type  $(r, s)$  on  $TM$   $2^{r+s}$   $M$ -tensors of the same type on  $TM$ , as explained in the following theorem for a tensor of type  $(1, 2)$ .

**THEOREM 4.2.** *Let  $S$  be a tensor of type  $(1, 2)$  on  $TM$ . Suppose that it is expressed in frame components, so that on  $TU$ ,*

$$(4.8) \quad \begin{aligned} S &= S_{jk}^i D_i \otimes \omega^j \otimes \omega^k + \dots + S_{jk}^i D_i \otimes \omega^{\bar{j}} \otimes \omega^{\bar{k}} \\ &+ \dots + S_{jk}^i D_{\bar{i}} \otimes \omega^{\bar{j}} \otimes \omega^{\bar{k}}. \end{aligned}$$

*Then the geometric objects  $S_1, \dots, S_4, \dots, S_8$  on  $TM$  whose components are  $(S_1)_{jk}^i = S_{jk}^i, \dots, (S_4)_{jk}^i = S_{jk}^i, \dots, (S_8)_{jk}^i = S_{jk}^{\bar{i}}$  are all  $M$ -tensors of type  $(1, 2)$ .*

The  $M$ -tensor  $S_4$  is independent of the  $M$ -connection  $\Gamma$  whose adapted frame is used in the formulation of (4.8). In fact

$$S_{jk}^i = S(\omega^i, D_{\bar{j}}, D_{\bar{k}}) = S(dx^i, \partial/\partial y^j, \partial/\partial y^k)$$

so that for any  $M$ -connection  $\Gamma$  the set  $S_{jk}^i$  of the frame components of  $S$  is

the same as the corresponding set of ordinary components of  $S$ . We call  $S_4$  the  $M$ -tensor associated with  $S$  and denote it by  $A(S)$ . The associated  $M$ -tensors for tensors of other types are defined similarly. It is clear that the associated  $M$ -tensor  $c$  of a symmetric tensor  $G$  of type  $(0, 2)$  which appeared in §3 is precisely the  $M$ -tensor  $A(G)$  associated with  $G$  as defined above.

Let  $A = [A^i]$  be an  $M$ -vector. Then (by Theorem 3.3),  $A^i \partial/\partial y^i$  is a vector (field) on  $TM$ . This vector is obviously vertical; we call it the *vertical lift* of  $A$  and denote it by  $A^V$ . In particular, the vertical lift of the vector  $\partial/\partial x^i$  on  $U$  is the vector  $\partial/\partial y^i$  on  $TU$ . Thus, the vertical lift gives rise to a natural isomorphism from  $T_x M$  to  $V_{(x,y)}$  defined by  $(\partial/\partial x^i)_x \rightarrow (\partial/\partial y^i)_{(x,y)}$ .

Now let  $\Gamma$  be an  $M$ -connection on  $TM$ , and  $A$  an  $M$ -vector on  $TM$ . Then (cf. Wong and Mok [1, §5])  $[-\Gamma_A^A]$  is a vector field on  $TM$  which is horizontal with respect to  $\Gamma$ . We call it the *horizontal lift* of  $A$  and denote it by  $A^H$ . In particular, the horizontal lift of the vector  $\partial/\partial x^i$  on  $U$  is the vector  $D_i$  on  $TU$ . Thus, the horizontal lift gives rise to a natural isomorphism from  $T_x M$  to  $H_\Gamma(x, y)$  defined by  $(\partial/\partial x^i)_x \mapsto (D_i)_{(x,y)}$ .

**5. Symmetric tensors of type  $(0, 2)$  on  $TM$ .** Let  $G$  be a nonzero symmetric tensor of type  $(0, 2)$  on  $TM$  which at present is not assumed to be of full rank or even of the same rank everywhere on  $TM$ , and let  $\Gamma$  be an  $M$ -connection. Then as in Theorem 4.2 we can express  $G$  on  $TU$  as

$$G = a_{ij} \omega^i \otimes \omega^j + h_{ji} \omega^i \otimes \omega^{\bar{j}} + h_{ij} \omega^{\bar{i}} \otimes \omega^j + c_{ij} \omega^{\bar{i}} \otimes \omega^{\bar{j}},$$

where  $[\omega^a]$  is the coframe on  $TU$  adapted to  $\Gamma$  and  $a = [a_{ij}]$ ,  $h = [h_{ij}]$  and  $c = [c_{ij}]$  are  $M$ -tensors of type  $(0, 2)$  on  $TM$  of which  $a$  and  $c$  are symmetric and  $c$  is the  $M$ -tensor  $A(G)$  associated with  $G$  and is independent of the choice of  $\Gamma$ .

Relative to the  $M$ -connection  $\Gamma$ , the frame component matrix of  $G$  is

$$\begin{bmatrix} a & h^t \\ h & c \end{bmatrix}.$$

Using relation (4.5) between the frame components and the ordinary components of  $G$ , we can prove

**THEOREM 5.1.** *The most general symmetric tensor  $G$  of type  $(0, 2)$  on  $TM$  has a component matrix of the form*

$$(5.1) \quad G = \begin{bmatrix} a + \Gamma'c\Gamma + h^t\Gamma + \Gamma'h & \Gamma'c + h^t \\ c\Gamma + h & c \end{bmatrix},$$

where  $c = A(G)$  is the  $M$ -tensor associated with  $G$ ,  $\Gamma$  is an  $M$ -connection and  $a$  (symmetric) and  $h$  are  $M$ -tensors of type  $(0, 2)$  on  $TM$ . Relative to the  $M$ -connection  $\Gamma$ , the frame component matrix of  $G$  is



$$(5.2) \quad \begin{bmatrix} a & h' \\ h & c \end{bmatrix}.$$

We note that  $G$  in (5.1) can be uniquely expressed as the following sum of three symmetric tensors of type  $(0, 2)$  on  $TM$ :

$$\begin{bmatrix} a - c & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c + \Gamma'c\Gamma & \Gamma'c \\ c\Gamma & c \end{bmatrix} + \begin{bmatrix} h'\Gamma + \Gamma'h & h' \\ h & 0 \end{bmatrix}.$$

We shall call these three tensors the *vertical lift of the  $M$ -tensor  $a - c$* , the *diagonal lift of the  $M$ -tensor  $c$*  and the *symmetric lift of the  $M$ -tensor  $h$*  respectively. The diagonal lift and the symmetric lift (which depend on  $\Gamma$ ) will reappear in subsequent discussions.

Let  $\tilde{\Gamma}$  be another  $M$ -connection, and

$$\begin{bmatrix} \tilde{a} & \tilde{h}' \\ \tilde{h} & \tilde{c} \end{bmatrix}$$

the frame component of  $G$  relative to  $\tilde{\Gamma}$ . Then we have

$$(5.3) \quad \tilde{c} = c,$$

$$(5.4) \quad \tilde{h} = h - cT, \quad \tilde{a} = a + T'cT - h'T - T'h,$$

where  $T = \tilde{\Gamma} - \Gamma$ . These relations suggest that we may express a symmetric tensor of type  $(0, 2)$  on  $TM$  as a certain equivalence class. In fact, let  $c$  be a symmetric  $M$ -tensor of type  $(0, 2)$  on  $TM$ ,  $\mathfrak{T}$  the set of ordered pairs  $(h, a)$  of  $M$ -tensors of type  $(0, 2)$  of which  $a$  is symmetric, and  $\sim$  the relation in  $\mathfrak{T}$  defined by  $(\tilde{h}, \tilde{a}) \sim (h, a)$  if (5.4) is satisfied for some  $M$ -tensor  $T$  of type  $(1, 1)$ . Then  $\sim$  is an equivalence relation and there is a bijection between the set of symmetric tensors of type  $(0, 2)$  on  $TM$  and the set of ordered pairs  $(c, [h, a]_c)$ , where  $[h, a]_c$  is the equivalence class containing  $(h, a)$ .

**6. Symmetric  $(0, 2)$ -tensors on  $TM$ -geometric considerations.** In order to gain a deeper insight into the structure of a symmetric  $(0, 2)$ -tensor  $G$  on  $TM$ , let us study briefly the geometry associated with  $G$ . As a symmetric  $(0, 2)$ -tensor,  $G$  assigns to each point  $\sigma \in TM$  a symmetric bilinear form  $G_\sigma$  on  $T_\sigma(TM)$  by means of which "orthogonality" can be defined in  $T_\sigma(TM)$ . Thus, two planes  $A, B$  at  $\sigma$  are said to be *orthogonal* (relative to  $G$ ) if  $G_\sigma(X, Y) = 0$  for all the vectors  $X \in A$  and  $Y \in B$ . Let  $A^\perp$  be the subspace of  $T_\sigma(TM)$  consisting of all the vectors orthogonal to the plane  $A$ . Then  $\dim(A \cap A^\perp)$  is called the *nullity* of  $A$ . In particular, we say that  $A$  is *null* if its nullity is equal to  $\dim A$ , and *nonnull* if its nullity is zero. It is easy to see that  $A$  is null iff the restriction of  $G_\sigma$  to  $A$  is zero, and  $A$  is nonnull iff this restriction is nonsingular.

Let  $c = A(G)$  with component matrix  $[c_{ij}]$  be the  $M$ -tensor associated with

$G$ , and  $V_\sigma$  the  $n$ -plane of the vertical distribution  $V$  at the point  $\sigma \in TM$ . Then for any two vectors  $X, Y$  of  $V_\sigma$  with component matrices

$$\begin{bmatrix} 0 \\ X^i \end{bmatrix}, \quad \begin{bmatrix} 0 \\ Y^i \end{bmatrix},$$

we have

$$G_\sigma(X, Y) = X^i G_{ij} Y^j = c_{ij}(\sigma) X^i Y^j.$$

Therefore,  $c_\sigma$  is the restriction of  $G_\sigma$  to  $V_\sigma$ , or to put it simply,  $c$  is the restriction of  $G$  to  $V$ .

Suppose  $G$  is expressed in the form given in Theorem 5.1. Applying our discussions above to the  $n$ -planes  $V_\sigma$  and  $H_\Gamma(\sigma)$  of the vertical distribution  $V$  and the horizontal distribution  $H_\Gamma$ , we can prove

**THEOREM 6.1.** *Let  $G$  be a symmetric tensor of type  $(0, 2)$  and  $\Gamma$  an  $M$ -connection on  $TM$ . Suppose that the frame component matrix of  $G$  relative to  $\Gamma$  is*

$$\begin{bmatrix} a & h^i \\ h & c \end{bmatrix}.$$

*Then, at an arbitrarily fixed point  $\sigma \in TM$ ,*

$$\begin{aligned} \text{rank } c &= n - \text{nullity of } V_\sigma, \\ (6.1) \quad \text{rank } a &= n - \text{nullity of } H_\Gamma(\sigma), \\ \text{rank } h &= n - \dim(H_\Gamma(\sigma) \cap V_\sigma^\perp). \end{aligned}$$

**PROOF.** We shall only prove (6.1)<sub>3</sub> as the proof for (6.1)<sub>1</sub> and (6.2)<sub>2</sub> is similar. We recall that  $H_\Gamma(\sigma)$  and  $V_\sigma$  are spanned respectively by the vectors  $D_i(\sigma)$  and  $D_{\bar{i}}(\sigma)$  of the adapted frame, so that an arbitrary vector in  $H_\Gamma(\sigma)$  has frame component matrix of the form  $\begin{bmatrix} X \\ 0 \end{bmatrix}$ . Now this vector lies in  $V_\sigma^\perp$  iff

$$\begin{bmatrix} X^i & 0 \end{bmatrix} \begin{bmatrix} a & h^i \\ h & c \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = 0, \quad \text{i.e., } X^i h^i = 0.$$

It follows from this that  $\dim(H_\Gamma(\sigma) \cap V_\sigma^\perp) = n - \text{rank } h$ , which is (6.1)<sub>3</sub>.  $\square$

We note that the assignment  $\sigma \rightarrow V_\sigma^\perp$  is generally not a distribution on  $TM$  and that the nullity of  $V_\sigma$  (or, equivalently, the rank of  $c$ ) generally varies from point to point. In the next two sections, we shall consider in detail the two extreme cases, where the rank of  $c$  is everywhere  $n$  or everywhere zero.

**7. Symmetric  $(0, 2)$ -tensors with nonsingular associated  $M$ -tensor.** Let us now consider the case of a symmetric  $(0, 2)$ -tensor  $G$  whose associated  $M$ -tensor  $c = A(G)$  is everywhere nonsingular. First, it is easy to prove

**THEOREM 7.1.** *The following conditions on a symmetric tensor  $G$  of type  $(0, 2)$  on  $TM$  are equivalent:*

- (i) *The  $M$ -tensor  $A(G)$  is everywhere of full rank  $n$ ,*

- (ii)  $V$  is nonnull with respect to  $G$ ,  
 (iii) The assignment  $\sigma \rightarrow V_\sigma^\perp$  is a horizontal distribution on  $TM$  (which we denote by  $V^\perp$ ).

Suppose that  $G$  is a symmetric  $(0, 2)$ -tensor satisfying any one of the equivalent conditions in Theorem 7.1. Then there is a unique  $M$ -connection  $\Gamma$  such that  $H_\Gamma = V^\perp$ . Let  $[D_\beta]$  be the frame adapted to  $\Gamma$ . Then,  $G(D_i, D_j) = 0$ . It follows that the frame component matrix of  $G$  is of the form  $\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$ , where  $c = A(G)$  and  $a = [a_{ij}]$  is the symmetric  $M$ -tensor determined by  $a_{ij} = G(D_i, D_j)$  and is therefore the restriction of  $G$  to  $H_\Gamma = V^\perp$ . In terms of  $c$ ,  $\Gamma$ , and  $a$ , the component matrix of  $G$  is exactly that in (3.7). Hence, we have the following version of Theorem 3.1:

**THEOREM 7.2.** *The most general symmetric tensor  $G$  of type  $(0, 2)$  on  $TM$  whose associated  $M$ -tensor  $c = A(G)$  is of full rank everywhere has a component matrix of the form*

$$(7.1) \quad G = \begin{bmatrix} a + \Gamma'c\Gamma & \Gamma'c \\ c\Gamma & c \end{bmatrix},$$

where  $\Gamma$  is the (unique)  $M$ -connection such that  $H_\Gamma = V^\perp$  and  $a$  is the symmetric  $M$ -tensor of type  $(0, 2)$  which is the restriction of  $G$  to  $V^\perp$ . Relative to this  $\Gamma$ , the frame component matrix of  $G$  is

$$(7.2) \quad \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}.$$

Consequently, the rank of  $G$  is equal to  $n + \text{rank } a$ . In particular,  $G$  is a (pseudo-) Riemannian metric iff  $a$  is of rank  $n$  everywhere.

Since  $c$ ,  $\Gamma$ ,  $a$  in Theorem 7.2 are uniquely determined when  $G$  is given, a consequence of Theorem 7.2 is that there is a bijection between the set of symmetric tensors of type  $(0, 2)$  on  $TM$  whose associated  $M$ -tensor is everywhere nonsingular and the set of ordered triples  $(c, \Gamma, a)$  where  $c$  (nonsingular) and  $a$  are symmetric  $M$ -tensors of type  $(0, 2)$  on  $TM$  and  $\Gamma$  is an  $M$ -connection.

**REMARK.** The  $M$ -tensor  $a$  considered above is intrinsically associated with  $G$ . It is interesting to see how it can be expressed in terms of the  $M$ -tensors  $\tilde{c}$ ,  $\tilde{h}$  and  $\tilde{a}$  which appear in the frame component matrix of  $G$  relative to an arbitrarily chosen  $M$ -connection  $\tilde{\Gamma}$ . To do this, we put  $h = 0$  in (5.4) and eliminate  $T$ , obtaining  $a = \tilde{a} - \tilde{h}'\tilde{c}^{-1}\tilde{h}$ .

In the special case where  $a = c$ , the symmetric tensor  $G$  of type  $(0, 2)$  in Theorem 7.2 is the diagonal lift of the nonsingular symmetric  $M$ -tensor  $c$  to  $TM$ , namely

$$\begin{bmatrix} c + \Gamma'c\Gamma & \Gamma'c \\ c\Gamma & c \end{bmatrix},$$

which is characterized by the conditions that for any two vectors  $X = [X^i]$ ,  $Y = [Y^i]$  in  $M$ ,

$$G(X^H, Y^H) = G(X^V, Y^V) = c_{ij}X^iY^j, \quad G(X^H, Y^V) = 0.$$

The following are two well-known examples of Riemannian metrics  $G$  whose associated  $M$ -tensor is everywhere nonsingular. Both of them are diagonal lifts.

(a) Let  $g = [g_{ij}]$  be a Riemannian metric on  $M$  and  $\Gamma_j^i = \{\Gamma_{jk}^i\}y^k$ , where  $\{\Gamma_{jk}^i\}$  is the Christoffel symbol of  $g_{ij}$ . Then  $\Gamma = [\Gamma_j^i]$  is an  $M$ -connection on  $TM$ . Using this  $\Gamma$  and putting  $a = c = g$  in (7.1), we get the component matrix of the original Sasaki metric (2.5) on  $TM$ .

(b) Let  $M$  be a Finslerian manifold and  $F: {}^sTM \rightarrow R$  the fundamental function defining the Finsler metric. It is easy to see that  $g_{ij} = \partial^2(\frac{1}{2}F^2)/\partial y^i\partial y^j$  are the components of a symmetric  $M$ -tensor of type  $(0, 2)$  on  ${}^sTM$  (which is everywhere nonsingular by definition of the fundamental function). If we define the functions  $\{\Gamma_{jk}^i\}$  and  $\Gamma_j^i$  on  ${}^sTU$  by

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2} g^{ih} (\partial_j g_{kh} + \partial_k g_{jh} - \partial_h g_{jk}),$$

$$\Gamma_j^i = \frac{1}{2} \frac{\partial}{\partial y^j} \left( \left\{ \begin{matrix} i \\ hk \end{matrix} \right\} y^h y^k \right),$$

then we can prove that  $\Gamma = [\Gamma_j^i]$  is an  $M$ -connection on  ${}^sTM$ . Using this  $\Gamma$  and putting  $a = c = g = [g_{ij}]$  in (7.1), we get the Riemannian metric on the slit tangent bundle of a Finslerian manifold considered by Yano and Davies in [1] and [2].

**8. Symmetric  $(0, 2)$ -tensors with zero associated  $M$ -tensor.** First, we can easily prove

**THEOREM 8.1.** *The following conditions on a symmetric tensor  $G$  of type  $(0, 2)$  on  $TM$  are equivalent:*

- (i) *The  $M$ -tensor  $A(G)$  is everywhere zero,*
- (ii)  *$V$  is null with respect to  $G$ ,*
- (iii)  *$V_\sigma \subset V_\sigma^\perp$  for all  $\sigma \in TM$ .*

*In case  $G$  is a (pseudo)-Riemannian metric, then  $\dim V_\sigma^\perp = n$  and so (iii) means that  $V_\sigma = V_\sigma^\perp$  for all  $\sigma \in TM$ .*

Now, putting  $c = 0$  in Theorem 5.1, we get

**THEOREM 8.2.** *The most general symmetric tensor  $G$  of type  $(0, 2)$  on  $TM$  whose associated  $M$ -tensor  $A(G)$  is zero has a component matrix of the form*

$$(8.1) \quad \begin{bmatrix} a + h'\Gamma + \Gamma'h & h' \\ h & 0 \end{bmatrix},$$

where  $\Gamma$  is an  $M$ -connection and  $a$  (symmetric),  $h$  are  $M$ -tensors of type  $(0, 2)$  on  $TM$ .

Relative to the  $M$ -connection  $\Gamma$ , the frame component matrix of  $G$  is

$$(8.2) \quad \begin{bmatrix} a & h' \\ h & 0 \end{bmatrix}.$$

The  $M$ -tensor  $h$  in (8.1) and (8.2) is independent of the  $\Gamma$  chosen, and we can prove that there is a bijection between the set of symmetric tensors of type  $(0, 2)$  on  $TM$  whose associated  $M$ -tensor is zero and the set of ordered pairs  $(h, [a]_h)$ , where  $h$  is an  $M$ -tensor of type  $(0, 2)$  and  $[a]_h$  is the equivalence class of the following equivalence relation in the set of symmetric  $M$ -tensors of type  $(0, 2)$ :

$\tilde{a} \sim a$  if there exists an  $M$ -tensor  $T$  of type  $(1, 1)$  such that  $\tilde{a} = a - h'T - T'h$ .

If  $G$  is a (pseudo-)Riemannian metric on  $TM$ , we can further simplify the component matrix (8.1). In fact, we have

**THEOREM 8.3.** *The most general (pseudo-)Riemannian metric  $G$  on  $TM$  whose associated  $M$ -tensor is zero has a component matrix of the form*

$$(8.3) \quad G = \begin{bmatrix} h'\Gamma + \Gamma'h & h' \\ h & 0 \end{bmatrix},$$

where  $\Gamma$  is an  $M$ -connection and  $h$  is a nonsingular  $M$ -tensor of type  $(0, 2)$  on  $TM$ . Relative to  $\Gamma$ , the frame component matrix of  $G$  is

$$(8.4) \quad \begin{bmatrix} 0 & h' \\ h & 0 \end{bmatrix}.$$

For a given  $G$ ,  $h$  is uniquely determined, whereas  $\Gamma$  is not.  $\Gamma$  is characterized by the property that  $H_\Gamma$  is null with respect to  $G$ . If  $\tilde{\Gamma}$  is any  $M$ -connection, then (8.3) with  $\Gamma$  replaced by  $\tilde{\Gamma}$  represents the same metric  $G$  iff  $\tilde{\Gamma} = \Gamma + (h')^{-1}d$  for some skew-symmetric  $M$ -tensor  $d$  of type  $(0, 2)$  on  $TM$ .

**PROOF.** From Theorem 8.2, we know that  $G$  is of the form (8.1), with  $h$  everywhere nonsingular. Since  $a$  is symmetric, we may write  $a + h'\Gamma + \Gamma'h = (h'\Gamma + \frac{1}{2}a) + (h'\Gamma + \frac{1}{2}a)'$ . As  $h$  is everywhere nonsingular, there exists a unique  $\Gamma^*$  satisfying  $h'\Gamma^* = h'\Gamma + \frac{1}{2}a$ , and this  $\Gamma^*$  is an  $M$ -connection because  $\frac{1}{2}(h')^{-1}a$  is an  $M$ -tensor of type  $(1, 1)$ . On renaming  $\Gamma^*$  as  $\Gamma$ , we get the desired form (8.3) for  $G$ . Clearly,  $h$  is uniquely determined and independent of the choice of  $\Gamma$ . Now, let  $\tilde{\Gamma}$  be any  $M$ -connection such that

$$h'\tilde{\Gamma} + \tilde{\Gamma}'h = h'\Gamma + \Gamma'h, \quad \text{i.e.,} \quad h'(\tilde{\Gamma} - \Gamma) + (\tilde{\Gamma} - \Gamma)'h = 0.$$

This means that  $d \equiv h'(\tilde{\Gamma} - \Gamma)$  is a skew-symmetric  $M$ -tensor of type  $(0, 2)$ . Hence,  $\tilde{\Gamma} = \Gamma + (h')^{-1}d$  as required.  $\square$

It follows from Theorem 8.3 above that there is a bijection between the set of such (pseudo-)Riemannian metrics on  $TM$  and the set of ordered pairs  $(h, [\Gamma]_h)$  where  $h$  is a nonsingular  $M$ -tensor of type  $(0, 2)$  and  $[\Gamma]_h$  is the equivalence class of the following equivalence relation in the set of  $M$ -connections on  $TM$ :

$\tilde{\Gamma} \sim \Gamma$  if  $\tilde{\Gamma} = \Gamma + (h')^{-1}d$  for some skew-symmetric  $M$ -tensor  $d$  of type  $(0, 2)$ .

We note that (8.3) is just the component matrix of the symmetric lift of the  $M$ -tensor  $h$ . Furthermore, from (8.4), we easily see that the metric of Theorem 8.3 has signature  $n$ , i.e., its canonical form has  $n$  positive and  $n$  negative signs.

An important example of the metric  $G$  in Theorem 8.3 is the horizontal lift  $g^H$  of a (pseudo-)Riemannian metric  $g$  on  $M$  to  $TM$  considered in Yano and Ishihara [2]. This can be obtained from (8.3) by putting  $h = g$  and  $\Gamma = [\Gamma'_k] = [\gamma^j \gamma_{jk}^i]$ , where  $\gamma_{jk}^i$  are the components of a linear connection  $\gamma$  on  $M$ . The component matrix of  $g^H$  is therefore

$$(8.5) \quad \begin{bmatrix} g\Gamma + \Gamma'g & g \\ g & 0 \end{bmatrix}.$$

In the special case where  $\gamma$  is a metric connection of  $g$ ,

$$\nabla_k g_{ij} \equiv \partial_k g_{ij} - \gamma_{ki}^a g_{aj} - \gamma_{kj}^a g_{ia} = 0$$

and so  $g\Gamma + \Gamma'g = \partial g$ . Thus in this case, (8.5) becomes

$$\begin{bmatrix} \partial g & g \\ g & 0 \end{bmatrix},$$

which is the component matrix of the complete lift  $g^C$  of  $g$  to  $TM$  considered by Yano and Kobayashi in [1].

We can now easily prove

**THEOREM 8.4.** *Let  $g$  be a (pseudo-)Riemannian metric and  $\gamma$  a linear connection on  $M$ . Then  $\gamma$  is a metric connection of  $g$  iff the horizontal distribution associated with the  $M$ -connection  $\Gamma_k^i = \gamma^j \gamma_{jk}^i$  on  $TM$  is null with respect to the complete lift  $g^C$  on  $TM$ .*

**9. Structure of a tensor of type  $(1, 1)$  on  $TM$ .** In the remaining sections, we shall study the structure of a tensor  $F$  of type  $(1, 1)$  on  $TM$ . Following Theorem 4.2, we suppose that  $\Gamma$  is an  $M$ -connection and express  $F$  on  $TU$  as

$$F = a_j^i D_i \otimes \omega^j + f_j^i D_i \otimes \omega^{\bar{j}} + b_j^i D_{\bar{i}} \otimes \omega^j + c_j^i D_{\bar{i}} \otimes \omega^{\bar{j}},$$

where  $[D_\beta]$  and  $[\omega^\alpha]$  are the frame and coframe on  $TU$  adapted to  $\Gamma$ . Then  $a = [a_j^i]$ ,  $f = [f_j^i]$ ,  $b = [b_j^i]$  and  $c = [c_j^i]$  are  $M$ -tensors of type  $(1, 1)$  and  $f$  is the  $M$ -tensor  $A(F)$  associated with  $F$  which is independent of the choice of  $\Gamma$ .

Relative to the  $M$ -connection  $\Gamma$ , the frame component matrix of  $F$  is  $\begin{bmatrix} a & f \\ b & c \end{bmatrix}$ . Using the relation (4.7) between the frame components and the ordinary components of  $F$ , we can prove

**THEOREM 9.1.** *The most general tensor  $F$  of type  $(1, 1)$  on  $TM$  has a component matrix of the form*

$$(9.1) \quad F = \begin{bmatrix} a + f\Gamma & f \\ b + c\Gamma - \Gamma a - \Gamma f\Gamma & c - \Gamma f \end{bmatrix},$$

where  $f = A(F)$  is the associated  $M$ -tensor of  $F$ ,  $\Gamma$  is an  $M$ -connection and  $a, b, c$  are  $M$ -tensors of type  $(1, 1)$  on  $TM$ . Relative to the  $M$ -connection  $\Gamma$ , the frame component matrix of  $F$  is

$$(9.2) \quad \begin{bmatrix} a & f \\ b & c \end{bmatrix}.$$

Let  $\tilde{\Gamma}$  be another  $M$ -connection, and

$$\begin{bmatrix} \tilde{a} & \tilde{f} \\ \tilde{b} & \tilde{c} \end{bmatrix}$$

the frame component matrix of  $F$  relative to  $\tilde{\Gamma}$ . Then we have

$$(9.3) \quad \tilde{f} = f,$$

$$(9.4) \quad \begin{aligned} \tilde{a} &= a - fT, \\ \tilde{c} &= Tf + c, \\ \tilde{b} &= Ta - TfT + b - cT, \end{aligned}$$

where  $T = \tilde{\Gamma} - \Gamma$ . Now let  $f$  be an  $M$ -tensor of type  $(1, 1)$  on  $TM$ ,  $\mathfrak{S}$  the set of ordered triples  $(a, c, b)$  of  $M$ -tensors of type  $(1, 1)$  on  $TM$ , and  $\sim$  the relation in  $\mathfrak{S}$  defined by  $(\tilde{a}, \tilde{c}, \tilde{b}) \sim (a, c, b)$  if (9.4) is satisfied for some  $M$ -tensor  $T$  of type  $(1, 1)$ . Then  $\sim$  is an equivalence relation and there is a bijection between the set of tensors of type  $(1, 1)$  on  $TM$  and the set of ordered pairs  $(f, [a, c, b]_f)$ , where  $[a, c, b]_f$  is the equivalence class containing  $(a, c, b)$ .

**10. Two classes of tensors of type  $(1, 1)$  on  $TM$ .** As in the case of a tensor of type  $(0, 2)$  on  $TM$ , we shall now confine our attention to a tensor of type  $(1, 1)$  on  $TM$  whose associated  $M$ -tensor has full rank everywhere or is zero. We first consider the geometric meaning of these assumptions.

Let  $F$  be a tensor of type  $(1, 1)$  on  $TM$ . Then  $F$  assigns to each point  $\sigma \in TM$  a linear transformation  $F_\sigma$  in  $T_\sigma(TM)$ . This  $F_\sigma$  maps the vertical  $n$ -plane  $V_\sigma$  at  $\sigma$  onto the subspace  $F_\sigma(V_\sigma)$ . To find  $F_\sigma(V_\sigma)$  we use the component matrix (9.1) of  $F$  and the component matrix  $[I]_\sigma^0$  of the  $n$  vectors

$D_j = \partial/\partial y^j$  which span  $V_\sigma$ . Since

$$\begin{bmatrix} a + f\Gamma & f \\ b + c\Gamma - \Gamma a - \Gamma f\Gamma & c - \Gamma f \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} f \\ c - \Gamma f \end{bmatrix},$$

we see that  $F_\sigma(V_\sigma)$  is spanned by the  $n$  vectors  $[c - \Gamma f]_\sigma$ . Thus,  $\dim F_\sigma(V_\sigma)$  in general varies from point to point.

Suppose  $f = A(F)$  is of full rank everywhere. We then have

$$\begin{bmatrix} f \\ c - \Gamma f \end{bmatrix} = \begin{bmatrix} I \\ cf^{-1} - \Gamma \end{bmatrix} f.$$

Since  $cf^{-1}$  is an  $M$ -tensor of type  $(1, 1)$ ,  $\tilde{\Gamma} = \Gamma - cf^{-1}$  is an  $M$ -connection. Therefore,  $F_\sigma(V_\sigma)$  is spanned by the  $n$  vectors  $[\tilde{\Gamma}^\Gamma_\sigma]$  and so, by (4.3),  $F_\sigma(V_\sigma) = H_{\tilde{\Gamma}}(\sigma)$ . Hence, the assignment  $\sigma \rightarrow F_\sigma(V_\sigma)$  is a horizontal distribution. Conversely, if this is true, then the matrix

$$\begin{bmatrix} f & 0 \\ c - \Gamma f & I \end{bmatrix}$$

must have rank  $2n$  and so  $f$  must have rank  $n$ .

On the other hand, if  $f = A(F) = 0$ , then  $F_\sigma(V_\sigma)$  is spanned by the  $n$  vectors  $[c^\Gamma_\sigma]$  and so  $F_\sigma(V_\sigma) \subset V_\sigma$ , i.e.,  $V_\sigma$  is stable under  $F_\sigma$ . Conversely, it is easy to see that if  $V_\sigma$  is stable under  $F_\sigma$  for every  $\sigma$ , then  $f = 0$ .

We summarize our discussion so far in

**THEOREM 10.1.** *Let  $F$  be a tensor of type  $(1, 1)$  on  $TM$  and  $A(F)$  its associated  $M$ -tensor. Then  $A(F)$  has full rank everywhere iff the assignment  $\sigma \rightarrow F_\sigma(V_\sigma)$  is a horizontal distribution  $F(V)$  on  $TM$ . And  $A(F) = 0$  iff  $V$  is stable under  $F$ .*

We now consider the case where  $A(F)$  has full rank everywhere. In this case we have the horizontal distribution  $F(V)$  and so it is natural to consider the frame component of  $F$  relative to the (unique)  $M$ -connection  $\Gamma$  such that  $H_\Gamma = F(V)$ . Now putting  $\tilde{\Gamma} = \Gamma$  in  $\tilde{\Gamma} = \Gamma - cf^{-1}$ , we get  $c = 0$ . Therefore, the frame component matrix of  $F$  relative to this  $\Gamma$  is of the form  $\begin{bmatrix} a & f \\ b & 0 \end{bmatrix}$ , where  $a, b$  are  $M$ -tensors of type  $(1, 1)$ . The component matrix of  $F$  can be obtained from Theorem 9.1. Thus, we have

**THEOREM 10.2.** *The most general tensor  $F$  of type  $(1, 1)$  on  $TM$  whose associated  $M$ -tensor  $f = A(F)$  has full rank everywhere has a component matrix of the form*

$$(10.1) \quad \begin{bmatrix} a + f\Gamma & f \\ b - \Gamma a - \Gamma f\Gamma & -\Gamma f \end{bmatrix},$$

where  $\Gamma$  is the  $M$ -connection such that  $H_\Gamma = F(V)$  and  $a, b$  are  $M$ -tensors of type  $(1, 1)$  on  $TM$ . Relative to the  $M$ -connection  $\Gamma$ , the frame component matrix



of  $F$  is

$$(10.2) \quad \begin{bmatrix} a & f \\ b & 0 \end{bmatrix}.$$

We remark that Theorem 10.2 can also be proved by using the fact that  $f$  is of rank  $n$  to simplify the component matrix in (9.1).

It follows from (10.1) that when  $F$  is given, first  $f = A(F)$ , then  $\Gamma$ , and then  $a$ , and finally  $b$  are uniquely determined. Therefore, there is a bijection between the set of tensors of type  $(1, 1)$  on  $TM$  whose associated  $M$ -tensor is everywhere nonsingular and the set of ordered quadruples  $(f, \Gamma, a, b)$  where  $f$  (nonsingular),  $a, b$  are  $M$ -tensors of type  $(1, 1)$  and  $\Gamma$  is an  $M$ -connection.

For the case  $A(F) = 0$ , we put  $f = 0$  in Theorem 9.1, and obtain

**THEOREM 10.3.** *The most general tensor  $F$  of type  $(1, 1)$  on  $TM$  whose associated  $M$ -tensor  $A(F)$  is zero has a component matrix of the form*

$$(10.3) \quad \begin{bmatrix} a & 0 \\ b + c\Gamma - \Gamma a & c \end{bmatrix},$$

where  $\Gamma$  is an  $M$ -connection and  $a, b, c$  are  $M$ -tensors of type  $(1, 1)$  on  $TM$ . Relative to the  $M$ -connection  $\Gamma$ , the frame component matrix of  $F$  is

$$(10.4) \quad \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}.$$

The  $M$ -tensors  $a$  and  $c$  in (10.4) are independent of the  $\Gamma$  chosen, and we can prove (cf. (9.4)) that there is a bijection between the set of such tensors on  $TM$  and the set of ordered triples  $(a, c, [b]_{(a,c)})$ , where  $a, c$  are  $M$ -tensors of type  $(1, 1)$  on  $TM$  and  $[b]_{(a,c)}$  is the equivalence class of the following equivalence relation in the set of  $M$ -tensors of type  $(1, 1)$ :

$\tilde{b} \sim b$  if there exists an  $M$ -tensor  $T$  of type  $(1, 1)$  on  $TM$  such that  $\tilde{b} = b + Ta - cT$ .

**11.  $(1, 1)$ -tensors  $F$  on  $TM$  satisfying  $F^2 = \lambda E$ .** Let  $F$  be a tensor of type  $(1, 1)$  on  $TM$  satisfying the condition  $F^2 = \lambda E$ , where  $\lambda$  is a real number and  $E$  is the "identity" tensor of type  $(1, 1)$  on  $TM$  with component matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . If  $\lambda = 1$ , then  $F$  is an almost product structure on  $TM$ . If  $\lambda = -1$ , then  $F$  is an almost complex structure on  $TM$ . If  $\lambda = 0$  and  $F$  is of rank  $n$ , then  $F$  is an almost tangent structure on  $TM$  (Clark and Bruckheimer [1]). In this section, we shall study such tensors  $F$  whose associated  $M$ -tensors have rank  $n$  everywhere or are zero, so that the results of §10 apply.

We first prove

**THEOREM 11.1.** (a) *Let  $F$  be any tensor of type  $(1, 1)$  on  $TM$  whose associated  $M$ -tensor  $f$  is everywhere nonsingular and which satisfies the condition  $F^2 = \lambda E$ . Then, there exists a unique  $M$ -connection  $\Gamma$  relative to which the frame*

component matrix of  $F$  is

$$(11.1) \quad \begin{bmatrix} 0 & f \\ \lambda f^{-1} & 0 \end{bmatrix}.$$

(b) If  $\Gamma$  is any  $M$ -connection,  $f$  any nonsingular  $M$ -tensor of type  $(1, 1)$  on  $TM$  and  $\lambda$  any real number, then (11.1) is the frame component matrix (relative to  $\Gamma$ ) of a tensor  $F$  of type  $(1, 1)$  on  $TM$  which satisfies the conditions that  $A(F) = f$  and  $F^2 = \lambda E$ .

PROOF. (a) By Theorem 10.2, there is an  $M$ -connection  $\Gamma$  relative to which the frame component matrix of  $F$  satisfies  $F^2 = \lambda E$ , then

$$\begin{bmatrix} a & f \\ b & 0 \end{bmatrix} \begin{bmatrix} a & f \\ b & 0 \end{bmatrix} = \lambda \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} a^2 + fb & af \\ ba & bf \end{bmatrix} = \lambda \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Therefore,  $a = 0$ ,  $b = \lambda f^{-1}$  and hence we have (11.1). To prove the uniqueness of  $\Gamma$ , we observe from Theorem 10.2 that if the frame component matrix of  $F$  relative to  $\Gamma$  is (11.1), then the component matrix of  $F$  is

$$F = \begin{bmatrix} f\Gamma & f \\ \lambda f^{-1} - \Gamma f\Gamma & -\Gamma f \end{bmatrix}.$$

Therefore, since  $f$  is everywhere nonsingular,  $\Gamma$  is uniquely determined. Hence (a) is completely proved. The proof of (b) is straightforward.  $\square$

Since linear connections exist on  $M$  and consequently  $M$ -connections exist on  $TM$ , if we take  $f = +I$  or  $-I$  in Theorem 11.1 (b), we have

**COROLLARY 11.2.** *There always exists on  $TM$  a tensor  $F$  of type  $(1,1)$  whose associated  $M$ -tensor is everywhere nonsingular and which satisfies the condition  $F^2 = \lambda E$ .*

In particular, if  $\lambda = -1$ ,  $f = -I$  and  $\Gamma$  is an  $h(1)$   $M$ -connection on  $TM$  (see Wong and Mok [1] for definition), we have the almost complex structure considered in Yano and Ishihara [1, §4].

As an example of a class of almost product structures on  $TM$ , we prove the following theorem on a pair of complementary horizontal distributions on  $TM$ .

**THEOREM 11.3.** *Let  $\Gamma^0, \Gamma$  be two  $M$ -connections on  $TM$ . Then  $H_{\Gamma^0}$  and  $H_{\Gamma}$  are complementary iff the  $M$ -tensor  $S = \Gamma - \Gamma^0$  of type  $(1, 1)$  on  $TM$  has full rank everywhere. In this case, the almost product structure  $F$  determined by  $H_{\Gamma^0}$*

and  $H_{\Gamma}$  is such that  $A(F) = 2S^{-1}$ , and, with respect to the  $M$ -connection  $\tilde{\Gamma} = \frac{1}{2}(\Gamma^0 + \Gamma)$ , the frame component matrix of  $F$  is

$$(11.2) \quad \begin{bmatrix} 0 & 2S^{-1} \\ \frac{1}{2}S & 0 \end{bmatrix},$$

which is (11.1) with  $f = 2S^{-1}$  and  $\lambda = 1$ .

PROOF. It suffices to see what happens in  $TU$ . In  $TU$ ,  $H_{\Gamma^0}$  and  $H_{\Gamma}$  are spanned by the vectors

$$[D_j^0] = \begin{bmatrix} I \\ -\Gamma^0 \end{bmatrix} \quad \text{and} \quad [D_j] = \begin{bmatrix} I \\ -\Gamma \end{bmatrix}$$

respectively. Therefore, a necessary and sufficient condition for  $H_{\Gamma^0}$  and  $H_{\Gamma}$  to be complementary is that the matrix

$$\begin{bmatrix} I & I \\ -\Gamma^0 & -\Gamma \end{bmatrix}$$

is of rank  $2n$  everywhere. But this matrix has the same rank as the matrix

$$\begin{bmatrix} 0 & I \\ \Gamma - \Gamma^0 & \Gamma \end{bmatrix},$$

and hence is of rank  $2n$  everywhere iff the  $M$ -tensor  $S = \Gamma - \Gamma^0$  is of rank  $n$  everywhere. Suppose now that  $H_{\Gamma^0}$ ,  $H_{\Gamma}$  are complementary. Then the almost product structure  $F$  determined by  $(H_{\Gamma^0}, H_{\Gamma})$  is the "reflection" about  $H_{\Gamma^0}$  along  $H_{\Gamma}$ , and is therefore determined by  $F(D_i^0) = D_i^0$ ,  $F(D_i) = -D_i$ . Applying  $F$  to the relation  $D_i - D_i^0 = -S_i^j D_j$  (where  $S = [S_i^j]$  and  $[D_j] = \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}$ ), we get

$$F(D_j) = 2(S^{-1})_j^i D_i^0 - D_j.$$

From this, it follows that the frame component matrix of  $F$  relative to  $\Gamma^0$  is

$$\begin{bmatrix} I & 2S^{-1} \\ 0 & -I \end{bmatrix}.$$

Since  $A(F) = 2S^{-1}$  is of full rank everywhere, there is (by Theorem 11.1) a unique  $M$ -connection  $\tilde{\Gamma}$  relative to which the frame component matrix of  $F$  is of the form (11.1). Since (11.2) is of the form (11.1), it must be the frame component matrix of  $F$  relative to  $\tilde{\Gamma}$ . To find  $\tilde{\Gamma}$ , we use (9.4)<sub>1</sub>, namely,  $\tilde{a} = a - fT$ , and substitute in it  $\tilde{a} = 0$ ,  $a = I$ ,  $f = 2S^{-1}$  and  $T = \tilde{\Gamma} - \Gamma^0$ . Then we get  $\tilde{\Gamma} - \Gamma^0 = \frac{1}{2}S = \frac{1}{2}(\Gamma - \Gamma^0)$ , and therefore,  $\tilde{\Gamma} = \frac{1}{2}(\Gamma^0 + \Gamma)$ .  $\square$

The proof of the next theorem is similar to that of Theorem 11.1.

**THEOREM 11.4.** (a) Let  $F$  be any tensor of type  $(1, 1)$  on  $TM$  whose associated

*M*-tensor  $A(F)$  is zero and which satisfies the condition  $F^2 = \lambda E$ . Relative to any *M*-connection  $\Gamma$ , the frame component matrix of  $F$  is of the form

$$(11.3) \quad \begin{bmatrix} a & 0 \\ b & c \end{bmatrix},$$

where  $a, b, c$  are *M*-tensors of type  $(1, 1)$  satisfying the conditions

$$(11.4) \quad a^2 = c^2 = \lambda I, \quad ba + cb = 0,$$

and  $a$  and  $c$  do not depend on  $\Gamma$  but  $b$  does.

(b) Conversely, if  $a, b, c$  are *M*-tensors of type  $(1, 1)$  on *TM* satisfying conditions (11.4), and  $\Gamma$  is any *M*-connection on *TM*, then (11.3) is the frame component matrix of a tensor  $F$  of type  $(1, 1)$  on *TM* satisfying the condition  $A(F) = 0$  and  $F^2 = \lambda E$ .

We now prove

**COROLLARY 11.5.** *There exists on *TM* a tensor  $F$  of type  $(1, 1)$  whose associated *M*-tensor is zero and which satisfies the condition  $F^2 = \lambda E$  iff there exists on *M* a tensor  $\alpha$  of type  $(1, 1)$  which satisfies the condition  $\alpha^2 = \lambda I$ . In particular, if  $\dim M = \text{odd}$ , there does not exist on *TM* any almost complex structure whose associated *M*-tensor is zero.*

**PROOF.** The “necessity” follows immediately from Theorem 11.4(a) and Theorem 3.2(e). To prove the “sufficiency” we put  $a = c = \alpha$  and  $b = 0$  in Theorem 11.4(b) and obtain a tensor  $F$  of type  $(1, 1)$  on *TM* whose frame component matrix is  $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$ . (This  $F$  is precisely the  $\alpha^H$  described in Example 1 below.) The last assertion in the corollary follows from the fact that if  $n$  is odd, there does not exist any real  $n \times n$  matrix  $A$  with  $A^2 = -I$ .  $\square$

Let us look at some examples of the tensor  $F$  in Theorem 11.4.

**EXAMPLE 1.** Let  $h$  be an almost complex structure on *M* and  $h^H$  its horizontal lift (Yano and Ishihara [2]). Then  $h^H$  is an almost complex structure on *TM* which can be obtained from Theorem 11.4 by putting  $a = c = h$ ,  $b = 0$  and  $\Gamma = [\Gamma_k^i] = [\gamma^j \gamma_{jk}^i]$ , where  $\gamma_{jk}^i$  are the components of a linear connection  $\gamma$  on *M*. The component matrix of  $h^H$  is therefore (cf. Theorem 10.3)

$$(11.5) \quad \begin{bmatrix} h & 0 \\ -\Gamma h + h\Gamma & h \end{bmatrix}.$$

In case  $\gamma$  is a linear connection with respect to which  $h$  is parallel,

$$\nabla_k h_j^i \equiv \partial_k h_j^i + \gamma_{ka}^i h_j^a - \gamma_{kj}^a h_a^i = 0$$

and so  $-\Gamma h + h\Gamma = \partial h$ . Therefore, in this case, (11.5) becomes

$$\begin{bmatrix} h & 0 \\ \partial h & h \end{bmatrix},$$

which is the component matrix of the complete lift  $h^C$  of  $h$  to  $TM$  as considered in Yano and Kobayashi [1].

EXAMPLE 2. Let  $\Gamma$  be an  $M$ -connection on  $TM$ . Then the "reflection" about  $H_\Gamma$  along the vertical distribution  $V$  is an almost product structure on  $TM$  whose frame component matrix relative to  $\Gamma$  and component matrix are respectively

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \begin{bmatrix} I & 0 \\ -2\Gamma & -I \end{bmatrix}.$$

This almost product structure has been studied in some detail by Grifone [1, § I-4] under the name of "nonhomogeneous connection".

EXAMPLE 3. Let  $b$  be an  $M$ -tensor of type  $(1, 1)$  on  $TM$  which is everywhere of full rank, and  $F$  the tensor of type  $(1, 1)$  on  $TM$  whose component matrix is  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  (which by Theorem 10.2 is also the frame component matrix of  $F$  relative to an arbitrary  $M$ -connection on  $TM$ ). Since  $F^2 = 0$ ,  $F$  is an almost tangent structure on  $TM$ . If  $b = I$ , we have the almost tangent structure considered in Clark and Goel [1, Example 1-1].

## 12. Metrics compatible with an almost product or almost complex structure.

Let  $G$  be a nonzero symmetric (resp. skew-symmetric) tensor of type  $(0, 2)$  and  $F$  a nonzero tensor of type  $(1, 1)$  on  $TM$ . Then, at each point  $\sigma$  of  $TM$ ,  $G$  induces an inner product (resp. exterior product)  $G_\sigma$  in  $T_\sigma(TM)$  while  $F$  induces a linear transformation  $F_\sigma$  in  $T_\sigma(TM)$ . We say that  $G$  is *compatible* with  $F$  (or  $G$  and  $F$  are *compatible*) if  $G_\sigma(F_\sigma X, F_\sigma Y) = G_\sigma(X, Y)$  for all  $X, Y \in T_\sigma(TM)$  and all  $\sigma \in TM$ . In terms of component matrices or frame component matrices,  $G$  is compatible with  $F$  iff  $F'GF = G$ .

Suppose now that  $G$  is compatible with  $F$  and  $F$  satisfies the condition  $F^2 = \lambda E$ , where  $\lambda$  is a real number, and  $E$  the "identity tensor" on  $TM$ . Then from

$$F'GF = G \Rightarrow F'(F'GF)F = F'GF = G$$

and  $F^2 = \lambda E$ , it follows that  $\lambda^2 G = G$  so that  $\lambda = \pm 1$ . Thus, if  $G$  is compatible with an  $F$  satisfying  $F^2 = \lambda E$ , then  $F$  must be an almost product or almost complex structure. For convenience, we shall refer to a tensor  $F$  of type  $(1, 1)$  on  $TM$  satisfying the condition  $F^2 = \lambda E$ , where  $\lambda = \pm 1$ , as a  $\lambda$ -structure on  $TM$ , and a nonsingular symmetric tensor of type  $(0, 2)$  on  $TM$  as a *metric* on  $TM$ .

In this section we determine all the metrics  $G$  and  $\lambda$ -structures  $F$  on  $TM$  whose associated  $M$ -tensors are everywhere nonsingular or zero and which are compatible with each other. Thus we shall always assume that  $\lambda = \pm 1$ . For convenience, we shall consider the following three cases separately:

Case 1.  $A(F)$  everywhere nonsingular,

Case 2a.  $A(F) = 0$  and  $A(G)$  everywhere nonsingular,

Case 2b.  $A(F) = 0$  and  $A(G) = 0$ .

*Case 1.  $f - A(F)$  everywhere nonsingular.* Since  $F$  satisfies  $F^2 = \lambda E$ , the frame component matrix of  $F$  relative to a suitably chosen  $M$ -connection  $\Gamma$  is

$$\begin{bmatrix} 0 & f \\ \lambda f^{-1} & 0 \end{bmatrix}$$

(cf. Theorem 11.1). Let  $G$  be a metric whose frame component matrix relative to  $\Gamma$  is

$$\begin{bmatrix} a & h' \\ h & c \end{bmatrix}.$$

Then the condition for  $G$  to be compatible with  $F$ , namely,

$$\begin{aligned} \begin{bmatrix} a & h' \\ h & c \end{bmatrix} &= \begin{bmatrix} 0 & \lambda(f')^{-1} \\ f' & 0 \end{bmatrix} \begin{bmatrix} a & h' \\ h & c \end{bmatrix} \begin{bmatrix} 0 & f \\ \lambda f^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} (f')^{-1} c f^{-1} & \lambda(f')^{-1} h f \\ \lambda f' h' f^{-1} & f' a f \end{bmatrix}, \end{aligned}$$

is that

$$a = (f')^{-1} c f^{-1}, \quad c = f' a f, \quad h = \lambda f' h' f^{-1},$$

which is equivalent to

$$a = (f')^{-1} c f^{-1}, \quad d \equiv h f \text{ satisfies } d' = \lambda d.$$

Hence we have proved

**THEOREM 12.1.** *A metric  $G$  on  $TM$  is compatible with a  $\lambda$ -structure  $F$  on  $TM$  whose associated  $M$ -tensor  $f = A(F)$  is everywhere nonsingular iff there exists an  $M$ -connection relative to which the frame component matrices of  $F$  and  $G$  are respectively*

$$\begin{bmatrix} 0 & f \\ \lambda f^{-1} & 0 \end{bmatrix},$$

$$\begin{bmatrix} (f')^{-1} c f^{-1} & \lambda(f')^{-1} d \\ d f^{-1} & c \end{bmatrix} = \begin{bmatrix} 0 & \lambda(f')^{-1} \\ I & 0 \end{bmatrix} \begin{bmatrix} c & \lambda d \\ d & c \end{bmatrix} \begin{bmatrix} 0 & I \\ \lambda f^{-1} & 0 \end{bmatrix},$$

where  $c = A(G)$  and  $d$  is a symmetric or skew-symmetric  $M$ -tensor of type  $(0, 2)$  according as  $\lambda = 1$  or  $\lambda = -1$  such that

$$\begin{bmatrix} c & \lambda d \\ d & c \end{bmatrix}$$

is nonsingular.

In particular, if  $f = I$  and  $\lambda = -1$ , then the most general metric compatible with the almost complex structure

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

has a frame component matrix of the form

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix},$$

which is therefore the sum of the diagonal lift of  $c$  and the symmetric lift of  $d$  relative to the  $M$ -connection  $\Gamma$  (cf. §5). On putting  $d = 0$ , we get the known result that the Sasaki metric is compatible with the almost complex structure

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

(cf. Tachibana and Okumura [1] and Yano and Davies [1]).

**Case 2a.**  $A(F) = 0$  and  $A(G)$  everywhere nonsingular. In this case, there is a unique  $M$ -connection  $\Gamma$  relative to which  $G$  has a frame component matrix of the form  $\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$ , where  $c = A(G)$  is nonsingular and  $a$  is a nonsingular symmetric  $M$ -tensor of type  $(0, 2)$  (cf. Theorem 7.2). At the same time, relative to any  $M$ -connection (and in particular, relative to  $\Gamma$ ), the frame component matrix of  $F$  satisfying  $F^2 = \lambda E$  is of the form

$$\begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix},$$

where  $\alpha, \beta, \gamma$  are  $M$ -tensors of type  $(1, 1)$  satisfying

$$(12.1) \quad \alpha^2 = \lambda I, \quad \gamma^2 = \lambda I, \quad \beta\alpha + \gamma\beta = 0$$

(cf. Theorem 11.4). The condition for the compatibility of  $G$  and  $F$  is

$$\begin{aligned} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} &= \begin{bmatrix} \alpha' & \beta' \\ 0 & \gamma' \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix} \\ &= \begin{bmatrix} \alpha'a\alpha + \beta'c\beta & \beta'c\gamma \\ \gamma'c\beta & \gamma'c\gamma \end{bmatrix}. \end{aligned}$$

Since  $c$  and  $\gamma$  are both nonsingular, this condition is easily seen to be equivalent to  $\beta = 0$  and

$$(12.2) \quad a = \alpha'a\alpha, \quad c = \gamma'c\gamma.$$

Multiplying the two equations in (12.2) on the right by  $\alpha$  and  $\gamma$  respectively

and making use of (12.1), we get

$$a\alpha = \lambda(a\alpha)', \quad c\gamma = \lambda(c\gamma)'.$$

From these it follows that  $d_1 \equiv \lambda a\alpha$ ,  $d_2 \equiv \lambda c\gamma$  are  $M$ -tensors of type  $(0, 2)$  such that

$$(12.3) \quad d_1' = \lambda d_1, \quad d_2' = \lambda d_2; \quad a = d_1\alpha, \quad c = d_2\gamma.$$

On account of the last two equations, the two equations in (12.2) become

$$(12.4) \quad d_1 = \alpha'd_1\alpha, \quad d_2 = \gamma'd_2\gamma.$$

Finally, it can easily be verified that when  $a, c$  are defined as in (12.3), then (12.3) and (12.4) imply that  $a' = a$  and  $c' = c$ . Hence we have proved

**THEOREM 12.2.** *A metric  $G$  on  $TM$  with nonsingular  $A(G)$  is compatible with a  $\lambda$ -structure  $F$  with  $A(F) = 0$  iff there exists an  $M$ -connection relative to which the frame component matrices of  $F$  and  $G$  are respectively*

$$\begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix}, \quad \begin{bmatrix} d_1\alpha & 0 \\ 0 & d_2\gamma \end{bmatrix},$$

where  $\alpha, \gamma$  are  $M$ -tensors of type  $(1, 1)$  such that  $\alpha^2 = \lambda I$ ,  $\gamma^2 = \lambda I$ , and  $d_1$  and  $d_2$  are nonsingular  $M$ -tensors of type  $(0, 2)$  which are symmetric if  $\lambda = 1$  and skew-symmetric if  $\lambda = -1$  and which satisfy the conditions

$$\alpha'd_1\alpha = d_1, \quad \gamma'd_2\gamma = d_2.$$

We recall (Corollary 11.5) that if  $\dim M$  is odd, then there does not exist on  $TM$  any almost complex structure  $F$  with  $A(F) = 0$ . In this case, the problem of finding the metrics on  $TM$  compatible with  $F$  does not arise. Thus, when  $\dim M$  is odd Theorem 12.2 has a meaning only for  $\lambda = 1$ , i.e., for an almost product structure.

*Case 2b.*  $A(F) = 0$  and  $A(G) = 0$ . In this case, we can take

$$\begin{bmatrix} 0 & h' \\ h & 0 \end{bmatrix}$$

as the frame component matrix of  $G$  relative to some suitably chosen  $M$ -connection  $\Gamma$  (cf. Theorem 8.3) and

$$\begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix}$$

as the frame component matrix of  $F$  relative to the same  $M$ -connection  $\Gamma$ , where the  $M$ -tensors  $\alpha, \beta, \gamma$  are such that

$$\alpha^2 = \lambda I, \quad \gamma^2 = \lambda I, \quad \beta\alpha + \gamma\beta = 0.$$

The condition for the compatibility of  $G$  and  $F$  is



$$\begin{bmatrix} 0 & h' \\ h & 0 \end{bmatrix} = \begin{bmatrix} \alpha' & \beta' \\ 0 & \gamma' \end{bmatrix} \begin{bmatrix} 0 & h' \\ h & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix} \\ = \begin{bmatrix} \beta'h\alpha + \alpha'h'\beta & \alpha'h'\gamma \\ \gamma'h\alpha & 0 \end{bmatrix}.$$

On account of (12.1) and putting  $d \equiv \beta'h\alpha$ , we can easily see that  $d$  is an  $M$ -tensor of type  $(0, 2)$  and the above condition of compatibility is equivalent to

$$(12.5) \quad d' = -d, \quad \gamma' = \lambda h\alpha h^{-1}, \quad \beta' = \lambda d\alpha h^{-1}.$$

The  $\gamma$  given by (12.5) is such that  $(\gamma^2)' = (\lambda h\alpha h^{-1})(\lambda h\alpha h^{-1}) = \lambda I$ , and therefore satisfies  $(12.1)_2$ . On the other hand, using (12.5) and  $(12.1)_1$ , we see that  $(12.1)_3$  is equivalent to

$$0 = \alpha'\beta' + \beta'\gamma' = \alpha'\lambda d\alpha h^{-1} + (\lambda d\alpha h^{-1})(\lambda h\alpha h^{-1}) \\ = \lambda(\alpha'd\alpha + d)h^{-1},$$

i.e.,

$$\alpha'd\alpha + d = 0.$$

On account of this and  $(12.5)_1$ ,  $(12.5)_3$  becomes

$$\beta = \lambda(h^{-1})'\alpha'(-d) = \lambda(h^{-1})'\lambda d\alpha = (h^{-1})'d\alpha.$$

Hence we have proved

**THEOREM 12.3.** *A metric  $G$  on  $TM$  with  $A(G) = 0$  is compatible with a  $\lambda$ -structure  $F$  on  $TM$  with  $A(F) = 0$  iff there exists an  $M$ -connection relative to which the frame component matrices of  $F$  and  $G$  are respectively*

$$\begin{bmatrix} \alpha & 0 \\ (h^{-1})'d\alpha & \lambda(h\alpha h^{-1})' \end{bmatrix}, \quad \begin{bmatrix} 0 & h' \\ h & 0 \end{bmatrix},$$

where  $h$  is an  $M$ -tensor of type  $(0, 2)$  which is everywhere nonsingular,  $\alpha$  is an  $M$ -tensor of type  $(1, 1)$  such that  $\alpha^2 = \lambda I$ , and  $d$  is a skew-symmetric  $M$ -tensor of type  $(0, 2)$  such that  $\alpha'd\alpha = -d$ .

As in Theorem 12.2, if  $\dim M$  is odd, then Theorem 12.3 has a meaning only for  $\lambda = 1$ , i.e., for an almost product structure.

In Theorems 12.2 and 12.3, there appear the  $M$ -tensors  $d_1$ ,  $d_2$  and  $d$  of type  $(0, 2)$  satisfying certain conditions. The question naturally arises whether these  $M$ -tensors exist. We answer this question in the affirmative by showing that with a given  $M$ -tensor  $\alpha$  of type  $(1, 1)$  on  $TM$  satisfying  $\alpha^2 = \lambda I$  ( $\lambda = \pm 1$ ), there always exist symmetric and skew-symmetric  $M$ -tensors of type  $(0, 2)$  satisfying  $\alpha'd\alpha = d$  or  $\alpha'd\alpha = -d$ . In fact, let  $d^0$  be any symmetric or skew-symmetric tensor of type  $(0, 2)$  on  $M$ , and put  $d = \frac{1}{2}(d^0 \pm \alpha'd^0\alpha)$ .

Then

$$\alpha'd\alpha = \frac{1}{2}(\alpha'd^0\alpha \pm d^0) = \pm d.$$

Thus,  $d$  is an  $M$ -tensor of type  $(0, 2)$  on  $TM$  having the required properties.

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